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Research Article

Application of Green-Naghdi equations for a nonlinear stability of a tangential-velocity discontinuity in shallow water flow.

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Abstract

It is well known that the Kelvin-Helmholtz instability (KHI) is always unstable in the absence of wave, regardless of the strength of velocity difference. But in the present of sound wave, the KHI is sup-pressed if the velocity difference is equal or greater than √8 times to the velocity of sound wave. Since a shallow water flow has an analogy with a compressible gas flow in 2D, a linear stability of the KHI in a shallow water flow was considered by Bedenzkov and Pogutse to obtain the same condition √8 of the Froude number which defines by the ratio of the velocity difference to the traveling velocity of gravity wave. In this paper, we consider the nonlinear stability of the KHI in a shallow water flow by using the Green-Naghdi equations. We obtain the critical value of the ratio of the Froude number to make the inter-face stabilized is smaller than √8 given by others. This critical value is analysed by using both of the analytical method and the Sturm's theory.

Keywords: kelvin-helmholtz instability; tangential-velocity discontinuity; green-naghdi equations; nonlinear analyse; shallow water flow.

Introduction

The discontinuity of a tangential velocity in an incompressible fluid is well known as the Kelvin Helmholtz instability. The flow is necessarily unstable, regardless of the velocity difference [1, 2]. However, a surprising results was shown by Landau [3] that the compressibility can suppress the KHI. The flow is stable for a large velocity difference. A mathematical analogy of sound waves in a compressible fluid and gravity waves in a shallow water flow was also mentioned (see p. 322 [3]). However, the flow of a compressible fluid is stable for only two-dimensional (2D) flow but always unstable for three-dimensional flow [4]. Cairns [5] showed that the physical mechanism for instability lies in the fact that the wave on the interface has a negative energy. The unstable region is produced by a coalescence of positive and negative energy modes. Miles [6] and Ribner [7] were the first study of over-reflexion problem related to the transmission and reflexion of a sound waves at a vortex sheet separating by two regions of constant horizontal velocity U_1 and U_2. This problem was extended by Fejer [8] by including the effects of hydromagnetic, then McKenzie [9] included the effects due to buoyancy. The similar stability of tangential-velocity discontinuity in a shallow water in 2D was given by Bedenzkov and Pogutse [10] since a shallow water flow has an analogy with a compressible gas flow in 2D. The horizontal length scale is assumed much greater than the vertical length scale. The analogy of stability theory of compressible fluids with that of shallow water is limited to two dimensions because the hydrostatic balance is employed in the vertical direction and that we have to consider perturbations, with wavelength $\lambda \gg H$, where H is the depth of the fluid layer, depending only on the coordinates of the horizontal plane of the liquid layer (not on the depth coordinate z) [11, 12]. The interface between two regions of fluid is stabilised if the velocity difference U is equal or greater than √8 times to the velocity of gravity waves $c = \sqrt{gH}$ [13, 14], with g being the gravity acceleration and H being the depth of water. However, Bedenzkov and Pogutse considered the linear stability of discontinuity surface by ignoring the higher-order of dissipative gravity waves.

In this paper, we consider the nonlinear stability of a tangential-velocity discontinuity by using the Green-Naghdi equations (GN). The GN describe the wave propagation of fully nonlinear and weakly dispersive gravity waves on the fluid of finite depth. The GN are derived by using shallow water scaling asymptotics for a domain of small aspect ratio, however there is no restriction placed on the Froude number [15, 16, 17]. The consideration of nonlinear wave plays a vital role in predicting the shape of wave such as tsunami and tides [18]. We obtain the dispersion relation of wave frequency and other characteristics by enforcing boundary conditions at the interface of tangential-velocity discontinuity. The stability characteristic of an interface is reduced by solving the dispersion equation. Five roots of complex frequency ω are gained as a functions of the velocity discontinuity U, the traveling speed c and the order of qH with H being the depth of water and q being the wavenumber in the direction x of wave propagation. The resulting dispersion relation is calculated analytically and compared with that used the Sturm´s theory to find the number of real roots of a polynomial equation [19, 20]. Sec. 2 shows the formulation of the problem and the dispersion equation. In Sec. 3, we go into the stability characteristic by both solving the dispersion equation analytically and using the Sturm´s theory. A summary and conclusions is presented in the last section (Sec. 4).

Formulations and Dispersion Relation

We consider two-region flow which is moving in the region $y > 0$ and is at rest for $y < 0$, as shown in Figure 1. We consider only the discontinuity of tangential velocity in the direction x of wave propagation, the normal velocity in the y direction is still continuous for entirely region. We consider small perturbation (proportional to $e^{i(qx-\omega t)}$) over the surface of discontinuity, suppose that

$u(x,y,z,t) = U_0+\tilde{u}(x,y,z,t)$, $v(x,y,z,t) = \tilde{v}(x,y,z,t)$, $h(x,y,t) = H+\tilde{u}(x,y,t)$, (1)

Where

$$U_0 = \begin{cases} U, & y > 0 \\ 0, & y < 0. \end{cases} \quad (2)$$

By ignoring the higher order items, the irrotational Green-Naghdi equations are written as

$$\frac{D\tilde{h}}{Dt} + H\left(\frac{\partial \tilde{u}}{\partial x} + \frac{\partial \tilde{v}}{\partial y}\right) = 0,$$
$$\frac{D\tilde{u}}{Dt} + g\frac{\partial \tilde{h}}{\partial x} = \frac{H^2}{3}\frac{\partial}{\partial x}\frac{D}{Dt}\left(\frac{\partial \tilde{u}}{\partial x} + \frac{\partial \tilde{v}}{\partial y}\right)$$
$$\frac{D\tilde{v}}{Dt} + g\frac{\partial \tilde{h}}{\partial y} = \frac{H^2}{3}\frac{\partial}{\partial y}\frac{D}{Dt}\left(\frac{\partial \tilde{u}}{\partial x} + \frac{\partial \tilde{v}}{\partial y}\right), \quad (3)$$

In which, $D/Dt = \partial/\partial t + U_0\partial/\partial x$.

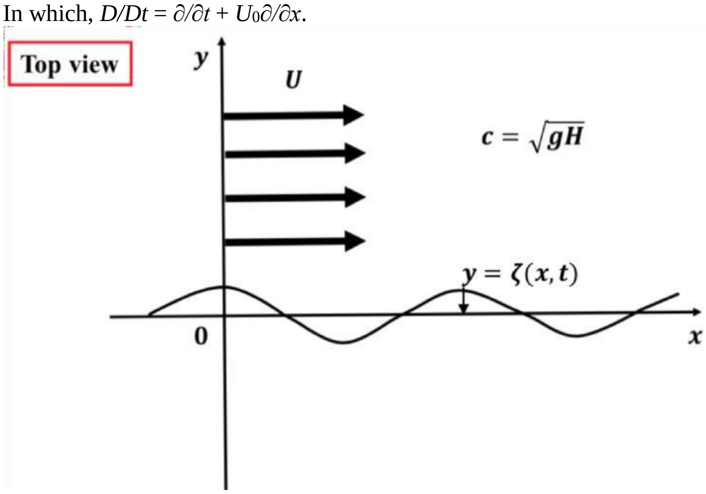

Figure 1: The top view of the interface of the tangential-velocity discontinuity in a shallow water flow. In region I ($y > 0$), the fluid is moving with uniform velocity U but is at rest in region II ($y < 0$). The surface of discontinuity in tangential velocity is horizontally perturbed to $y = \zeta(x,t)$ with an infinitesimal amplitude.

By taking derivative of two last equations on x, y respectively and then substituting into the first one, we obtain easily

$$\frac{D^2\tilde{h}}{Dt^2} - gH\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\tilde{h} - \frac{H^2}{3}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\frac{D^2\tilde{h}}{Dt^2} = 0. \quad (4)$$

We seek solution in the form $e^{i(qx-\omega t)}\, e^{\kappa y}$, where $\kappa = -\kappa_1$ for $y > 0$ and $\kappa = \kappa_2$ for $y < 0$. In region I ($y > 0$) with $U_0 = U$, therefore we have the relation between wavenumber κ_1 and other quantities as follows

$$\Big[-i(\omega-qU)\Big]^2 - gH(\kappa_1^2-q^2) + \frac{H^2}{3}(q^2-\kappa_1^2)\Big[-i(\omega-qU)\Big]^2 = 0.$$
$$\Longleftrightarrow \kappa_1^2 = \frac{q^2c^2 - (\omega-qU)^2\left(1+\frac{H^2q^2}{3}\right)}{c^2 - \frac{H^2}{3}(\omega-qU)^2}. \quad (5)$$

Similarly, in region II ($y < 0$) with $U_0 = 0$, we have

$$\kappa_2^2 = \frac{q^2c^2 - \omega^2\left(1+\frac{H^2q^2}{3}\right)}{c^2 - \frac{H^2}{3}\omega^2}. \quad (6)$$

The kinematic boundary condition is

$$\tilde{v} = \frac{\partial \tilde{\zeta}}{\partial t} + U_0\frac{\partial \tilde{\zeta}}{\partial x}, \text{ at } y = \tilde{\zeta}, \quad (7)$$

Where $\zeta(x,t) = ae^{i(qx-\omega t)}$ is the position of the discontinuity surface, with a is small constant.

The normal component of the velocity are equal on both sides of the interface and equal to the movement of the surface in that direction. Thus, this gives

$\tilde{v}_1 = \tilde{v}_2$ at $y = \zeta(x,t) \approx 0$. (8)

The pressure p should be continuous at the tangential discontinuity. In the hydrostatic approximation $p = -\rho g\tilde{h}$, the condition of pressure is reduced to that of the wave depth on both sides of the interface:

$\tilde{h}_1 = \tilde{h}_2$ at $y = \zeta(x,t) \approx 0$. (9)

The horizontal displacement ζ of the interface is connected with the vertical one $\tilde{h}$, thus we obtain

$$\frac{\kappa_1[c^2 - \frac{H^2}{3}(\omega-qU)^2]}{(\omega-qU)^2} = -\frac{\kappa_2[c^2 - \frac{H^2}{3}\omega^2]}{\omega^2}. \quad (10)$$

Combination of (5), (6) and (10), we reduce the dispersion relation between the wave frequency ω and other quantities as follows

$$\left[\omega^2 - (\omega-qU)^2\right]\left[q^2c^2[\omega^2+(\omega-qU)^2] - (1+\frac{2}{3}q^2H^2)(\omega-qU)^2\omega^2\right] = 0. \quad (11)$$

The first factor gives one real root $\omega = \frac{1}{2}qU$ which does not contribute to the instability of the interface. Thus, we do not consider this root longer. The instability now is considered simply from the contribution of the second factor as

$$q^2c^2[\omega^2 + (\omega-qU)^2] - (1+\frac{2}{3}q^2H^2)(\omega-qU)^2\omega^2 = 0. \quad (12)$$

Stability Analyze of an Interface

Analytical Solution

The interface is stabilized if and only if the dispersion equation (12) has only real roots, since it is a quartic polynomial of with real coefficients of argument ω. We introduce new variable as follows

$$\Omega = \omega + \frac{qU}{2}, \quad \textbf{(13)}$$

The dispersion equation (12) then turns to

$$q^2c^2\left[\left(\Omega+\frac{qU}{2}\right)^2+\left(\Omega-\frac{qU}{2}\right)^2\right]-\left[1+\frac{2}{3}q^2H^2\right]\left(\Omega-\frac{qU}{2}\right)^2\left(\Omega+\frac{qU}{2}\right)^2 = 0. \quad (14)$$

Equation (14) gives four analytical solutions as follow

$$\Omega_{\pm,\pm} = \pm\frac{1}{2}\sqrt{\frac{12q^2c^2+3q^2U^2+2H^2q^4U^2\pm 4\sqrt{3}\sqrt{c^2q^4(3c^2+3U^2+2H^2q^2U^2)}}{3+2H^2q^2}}$$
$$= \pm\frac{1}{2}qc\sqrt{\frac{12+3M^2+2H^2q^2M^2\pm 4\sqrt{3}\sqrt{3+3M^2+2H^2q^2M^2}}{3+2H^2q^2}} \quad (15)$$

The stabilized condition reduces to

$$12+3M^2+2H^2q^2M^2-4\sqrt{3}\sqrt{3+3M^2+2H^2q^2M^2}\geq 0$$
$$M^2(3+2H^2q^2)\Big(-24+M^2(3+2H^2q^2)\Big)\geq 0, \quad (16)$$

Where $M = U/c$ defines the Froude number, then we have

$$M^2 \geq 8\left(1+\frac{2}{3}H^2q^2\right)^{-1}, \quad (17)$$

Therefore, the interface is stabilised if and only if $M \geq \sqrt{8}(1+\frac{2}{3}q^2H^2)^{-\frac{1}{2}}$. This critical value of Froude number to make the interface stabilised is smaller than the one $\sqrt{8}$ given by Landau [3] and the others [4, 10].

Figure 2 displays the imaginary and real part roots given in (15) for a given $qH = 1$. The imaginary part of roots $\Omega_{\pm,-}$ tend to zero at the Froude number $M = \sqrt{24/5} \approx 2.19089 < 2.822 \approx \sqrt{8}$ and then vanish for greater values of the Froude number. Two other roots $\Omega_{\pm,+}$ are always real for any positive Froude number, *i. e.* the imaginary parts $\text{Im}[\Omega_{\pm,+}] = 0$. Therefore, the flow is stable if and only if the Froude number M is equal or greater than $\sqrt{24/5}$; otherwise unstable.

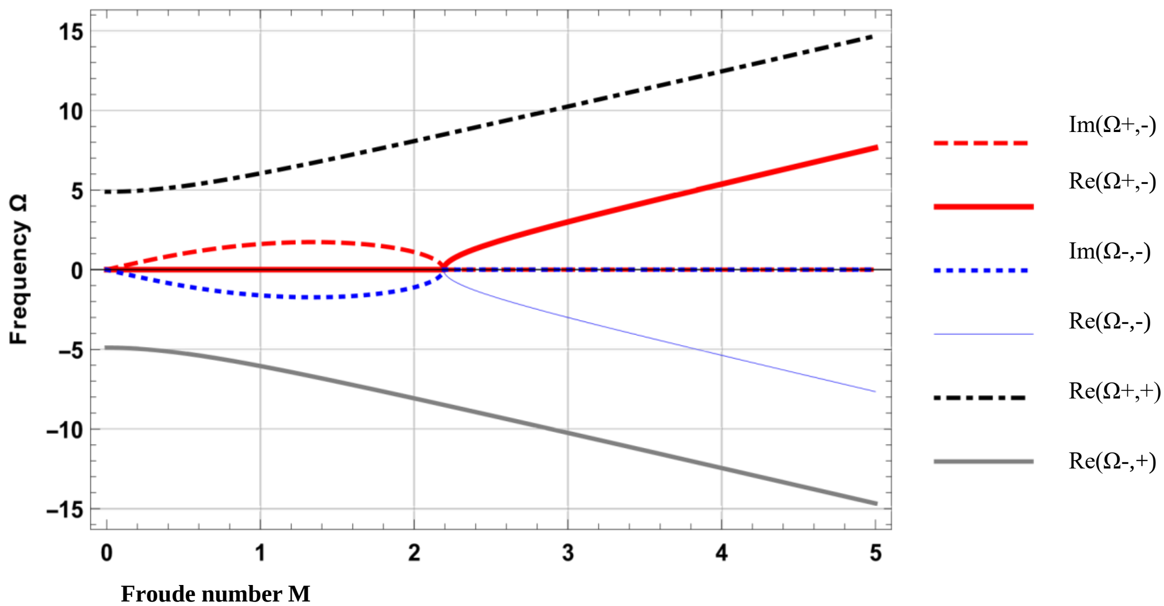

Figure 2: The imaginary and real parts of root Ω in equation (15) for a given $qH = 1$. Since two roots $\Omega_{\pm,+}$ are always real, their imaginary parts are zero.

By using the Sturm's Theorem

Next, we go to find the condition of the dispersion equation (12) to have all real roots by using the Sturm's Theorem (see appendix B) since the flow is stable if and only if (12) has only real roots. The Sturm's sequences are constructed as following:

$$p_0(\omega) = (1+\frac{2}{3}q^2H^2)(\omega-qU)^2\omega^2 - q^2c^2[\omega^2+(\omega-qU)^2]$$
$$p_1(\omega) = p_0'(\omega),$$

$p_2(\omega) = -\text{remainder}(p_0, p_1)$, (18)

$p_3(\omega) = -\text{remainder}(p_1, p_2)$,

$$p_4(\omega) = -\text{remainder}(p_2, p_3) = \frac{1}{48}q^4U^2\Big[(3+2q^2H^2)U^2-24c^2\Big],$$

Here remainder (p_j, p_{j+1}) is the remainder of the devision of polynomial p_j to p_{j+1} for $j = 1, ..., 3$. All roots are real if and only if all highest coefficients of $p_i(\omega)$, $i = 0,...,4$ must have same signs. After simple calculation, we find the highest coefficients of $p_0(\omega)$, $p_1(\omega)$, $p_2(\omega)$, $p_3(\omega)$ are always positive. Thus, we just need $p_4(\omega)$ positive, that is

$$(3+2q^2H^2)U^2-24c^2\geq 0$$
$$\Longleftrightarrow U \geq \sqrt{8}c(1+\frac{2}{3}q^2H^2)^{-\frac{1}{2}},$$
$$\Longleftrightarrow M \geq \sqrt{8}(1+\frac{2}{3}q^2H^2)^{-\frac{1}{2}}, \quad (19)$$

In which M is the Froude number defined in the section 3.1. – This result coincides with that given in (17). In other words, we found that the critical value of the Froude number M to make the interface stabilised is $\sqrt{8}(1+\frac{2}{3}q^2H^2)^{-\frac{1}{2}}$ which is smaller than that one $\sqrt{8}$ given by others.

Conclusion

We have considered the nonlinear stability of the Kelvin-Helmholtz instability in a shallow water flow by using the Green-Naghdi equations. The stability characteristic of the interface obtains by analysing the dispersion relation of wave frequency and others. The analytical solution is compared with that used the Sturm's theory. We obtain a coincidence of two these methods and show that the flow is stable with an amount of velocity discontinuity smaller than that given by previous researches [3, 4, 10].

(19)

The stability condition of flow depends on both the strength of velocity difference and the depth of considered region. For a given velocity difference, the flow is more stable in deeper regions and is less stable in shallower regions.
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A The irrotational Green-Naghdi equations

The Green Naghdi equations (GN) is the fully nonlinear shallow-water waves whose amplitude is not necessarily small and represents a higher-order correction to the classical shallow-water equations. In shallow-water approximation, the velocity field of three dimensional long wavelength propagation can approximate over the depth as follows [17]

$$u(x,y,z,t) \approx \bar{u}(x,y,t) = \frac{1}{h(x,y,t)} \int_0^h u(x,y,z,t)\,dz. \tag{20}$$

Then we can derive the irrotational Green-Naghdi equations [17]

$$\begin{cases} h_t + \nabla \cdot [h\bar{u}] = 0, \\ u_t + \nabla[|\bar{u}|^2/2] + g\nabla h + \nabla \cdot [h^3\gamma]/(3h) = \bar{u} \cdot \nabla h/3\nabla[h\nabla \cdot \bar{u}] - [\bar{u} \cdot \nabla(h\nabla \cdot \bar{u}/3)]\nabla h, \end{cases} \tag{21}$$

Where $\nabla = (\partial/\partial x, \partial/\partial y)$ is horizontal gradient; $\gamma = (\nabla\cdot\bar{u})^2 - \nabla\cdot\bar{u}_t - \bar{u}\cdot\nabla[\nabla\cdot\bar{u}]$ is the vertical acceleration at the free surface and can be derived as

$$\gamma = \frac{D}{Dt}\left(\frac{\partial \bar{u}}{\partial x} + \frac{\partial \bar{v}}{\partial y}\right) - \left(\frac{\partial \bar{u}}{\partial x} + \frac{\partial \bar{v}}{\partial y}\right)^2. \tag{22}$$

B Sturm's theorem

Here we introduce the Sturm's theorem [19, 20] which expresses the number of distinct real roots of a univariate polynomial *p* located in an interval in terms of the number of changes of signs of the values of the Sturm's sequence at the bounds of the interval. Applied to the interval of all the real numbers, it gives the total number of real roots of *p*. The **Sturm sequence** is a finite sequence of polynomials, applying Euclid's algorithm to *p* and its derivative:

$$\begin{aligned} p_0(x) &= p(x), \\ p_1(x) &= p_0'(x), \\ p_2(x) &= -\text{remainder}(p_0,p_1), \\ p_3(x) &= -\text{remainder}(p_1,p_2), \\ &\vdots \\ p_m &= -\text{remainder}(p_{m-2},p_{m-1}), \\ 0 &= -\text{remainder}(p_{m-1}, p_m). \end{aligned}$$

where remainder(p_j,p_{j+1}) is the remainder of the polynomial long division of p_j by p_{j+1}, and where *m* is the minimal number of polynomial divisions (never greater than *deg*(*p*)) needed to obtain a zero remainder.

Let $\sigma(\xi)$ denote the number of sign changes (ignoring zeroes) in the sequence $p_0(\xi),p_1(\xi),p_2(\xi),\cdots,p_m(\xi)$.

Sturm's theorem then states that for two real numbers $a < b$, the number of distinct real roots of *p* in the interval $[a,b]$ is $\sigma(a) - \sigma(b)$.

Choosing $a = -\infty$, $b = \infty$, then the total number of real roots of a polynomial is equal to $\sigma(-\infty) - \sigma(\infty)$. That is to say, all roots of a polynomial of degree *m* are real, if and only if $\sigma(-\infty) - \sigma(\infty) = m$.

As $0 \le \sigma(-\infty) \le m$, $0 \le \sigma(\infty) \le m$, so $\sigma(-\infty) = m, \sigma(\infty) = 0$.

Since the sign of a polynomial is decided by the term with highest degree as $x \to \pm\infty$. Thus all highest coefficients of the polynomials in the Sturm sequence must have the same signs (all positive or all negative).

In conclusion, we have the following Lemma: Suppose $p(x)$ is a univariate polynomial with real coefficients, *m* is the highest degree of $p(x)$. $p_0(\xi)$, $p_1(\xi)$, $p_2(\xi),\cdots,p_m(\xi)$ is the Sturm sequence of $p(x)$. If all highest coefficients of $p_i(x), i = 0,1,\cdots,m$ have the same signs (all positive or all negative), then all roots of the polynomial $p(x)$ are real.

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